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Symplectic dynamics on the universal Grassmannian

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Both finite- and infinite-dimensional integrable systems can be linearized on orbits of the infinite Abelian group Γ^+ on the universal Grassmannian. Our aim is to link these results to standard symplectic dynamics by giving an explicit Hamiltonian formulation on the symplectic manifold $M = \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})/\mathrm{Gl}(\mathcal{H}_+) \times \mathrm{Gl}(\mathcal{H}_-)$. We also construct a recursion operator for the action of $\mathrm{LU}(1)^+$, a real version of Γ^+, on the Grassmannian $\mathrm{U}_{\mathrm{res}}(\mathcal{H})/\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$.

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1. Introduction

All known completely integrable systems in both finite- and infinite-dimensional mechanics are actually algebraically integrable. In particular the invariant tori are "real parts" of Abelian varieties (see, e.g., ref. [9]) which happen to be the Jacobians of the so-called spectral curves. This phenomenon is basically due to the existence of a Lax description which links mechanics with the algebraic geometry of curves and their Jacobians. For infinite-dimensional systems, the pioneering work by McKean and Trubowitz, dealing with the spectrum of the Hill operator for KdV, yields an infinite-genus hyperelliptic spectral curve [8]. The finite-genus case has more a seed description, spelled out by several authors (see, e.g., ref. [6]).

Parallel to this, there is the set up started by Sato linking algebraic integrability to the geometry of an infinite Grassmannian manifold. In the following we will need the Hilbert–Schmidt model $\mathrm{Gr}(\mathcal{H})$ of this Grassmannian as worked out in ref. [10] and we refer to ref. [12] for the study of the KP hierarchy in this set up. The link between these two approaches is given by the Krichever map, which associates to a spectral curve and a point on its Jacobian (plus extra data) a point in $\mathrm{Gr}(\mathcal{H})$.

Unfortunately, in this algebraic setting the Hamiltonian structure is somewhat hidden. Basically one deals with a single complex torus which, although "generic", is kept fixed all the time. In this way one has a "universal" description of the angle variables, while actions are dealt with in concrete examples. The main purpose of this paper is to dig out the Hamiltonian structure in the Grassmannian approach with the hope of making it more appealing to physicists.

Recall that a Liouville integrable Hamiltonian system with Hamiltonian h is a differentiable map $f : V \to B$ of a $2m$-dimensional symplectic manifold (V, ω) onto an open set $B \subset \mathbb{R}^m$, such that the components of f are independent (i.e., $df_1 \wedge \cdots \wedge df_m \neq 0$), in involution (i.e., $\{f_i, f_j\} = 0$) and $h = r \circ f$ for some smooth function $r : B \to \mathbb{R}$. If f is proper, the connected components of $f^{-1}(b)$ are m-dimensional tori. If f is regular enough, there are action–angle coordinates $(I_1, ..., I_m, \phi_1, ..., \phi_m)$, the ϕ_i being determined mod 2π, such that $f_i = f_i(I_1, ..., I_m)$ and $\omega = \sum dI_i \wedge d\phi_i$. Such a system is called algebraically integrable whenever there is a smooth algebraic variety $\tilde{V}$, endowed with a closed non-degenerate holomorphic $(2,0)$-form $\tilde{\omega}$ and a function $\tilde{h} : \tilde{V} \to \mathbb{C}$, together with a proper surjective map $\tilde{f} : \tilde{V} \to \tilde{B}$, with $\tilde{B}$ open in $\mathbb{C}^m$, such that V is a component of the set of real points of $\tilde{V}$, $\omega = \tilde{\omega}|_V$, and h is a smooth function of $\tilde{f}|V$. It follows that the fibres of $\tilde{f}$ are (possibly degenerate limits of) Abelian varieties.

Whenever an Abelian variety appearing in connection with algebraically integrable systems is actually the Jacobian of a curve C, one can map it into $\mathrm{Gr}(\mathcal{H})$ via the Krichever map (for more details see, e.g., refs. [12,10]) and the Jacobian itself appears as an orbit of the subgroup $\Gamma^+ \subset \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$.

As is well known, $\mathrm{Gr}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})/\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ is a Kähler manifold, carrying strictly Hamiltonian action of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. This space is, however, too small to carry a Hamiltonian action of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ and in particular of the subgroup Γ^+ along whose orbits the KP flow is linearized. To construct an explicit bridge between symplectic mechanics and the Grassmannian approach, we enlarge $\mathrm{Gr}(\mathcal{H})$ to the manifold $M = \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})/\mathrm{Gl}(\mathcal{H}_+) \times \mathrm{Gl}(\mathcal{H}_-)$, on which the central extension of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ reduces to a symplectic form. This space is quite natural as the example of the harmonic oscillator shows (see appendix B). In this environment we can work out symplectic mechanics in a quite standard way and, as a byproduct, we can explain why one can actually project the flows on $\mathrm{Gr}(\mathcal{H})$ without losing information. As we will see, this is the same as projecting from the phase space to the configuration space, a place where Hamiltonian mechanics hardly lives. This projection is, however, costless from the algebraic point of view, since there is an isomorphism between the relevant tori in M and in $\mathrm{Gr}(\mathcal{H})$.

The manifold M is a "universal phase space" containing all (complexified) local phase spaces of algebraically integrable systems, in the sense that these appear as submanifolds of M. Here algebra plays a role; one easily gets recursion operators by simple algebraic operations which build up Abelian algebras of Hamiltonian vector fields on M with no further restrictions.

The last question we address is about recovering real phase spaces and dynamics. We give a real structure on M, whose real points precisely belong to $\mathrm{Gr}(\mathcal{H})$. On this set the dynamics is linearized along the flow of $\mathrm{LU}(1)^+$, a real version of Γ^+. We also give a general expression for the recurrence operator relative to this Abelian algebra.

Although linking to the enormous literature on integrable systems is beyond the size of this paper, there are several interesting questions one can ask and possibly answer in this set up. We will comment on these in the final section.

2. Constructing the "universal" phase space

To fix notations, we recall some basic definitions and refer to ref. [10] for details. Let $\mathcal{H} = \mathrm{L}^2(\mathrm{S}^1, \mathbb{C})$ be the space of all square-integrable complex valued functions on the circle $\mathrm{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. In $\mathcal{H}$ there is an orthonormal basis given by the functions $\{z^k, k \in \mathbb{Z}\}$ and a related orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with $\mathcal{H}_+$ and $\mathcal{H}_-$ the closed subspaces spanned by the elements $\{z^k\}$ with $k \geq 0$ and $k < 0$, respectively.

In the following we will be mainly interested in the subgroup $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ of the group $\mathrm{Gl}(\mathcal{H}) \subset B(\mathcal{H})$ of bounded invertible operators defined as follows. If we write any $g \in \mathrm{Gl}(\mathcal{H})$ in the block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, the group $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ is the subgroup of $\mathrm{Gl}(\mathcal{H})$ made of operators g whose off-diagonal blocks b and c are Hilbert–Schmidt. As g is invertible, the blocks a and d are automatically Fredholm. The Lie algebra of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$, denoted $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$, consists of all bounded operators A of the form (1) which are not necessarily invertible and with off-diagonal blocks b and c Hilbert-Schmidt. The algebra $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ is a Banach and a Banach–Lie algebra with respect to the norm $\|\cdot\|_J$ (see ref. [10], p. 80). This fact will be helpful in constructing recursion operators with vanishing Nijenhuis torsion as we shall see below. The restricted unitary group is $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}(\mathcal{H}) \cap \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$, whose Lie algebra $\mathrm{u}_{\mathrm{res}}(\mathcal{H})$ is made of anti-Hermitian elements of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$.

The next object we need is the Grassmannian $\mathrm{Gr}(\mathcal{H})$ of $\mathcal{H}$. This is the collection of all closed subspaces W of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ which are "comparable" with $\mathcal{H}_+$ in the sense that the orthogonal projection $\mathrm{pr}_+ : W \to \mathcal{H}_+$ is a Fredholm operator. To give $\mathrm{Gr}(\mathcal{H})$ the structure of a holomorphic Hilbert manifold, one also requires that the orthogonal projection $\mathrm{pr}_- : W \to \mathcal{H}_-$ is a Hilbert–Schmidt operator. As a result, the local model for $\mathrm{Gr}(\mathcal{H})$ is the space $\Im_2(\mathcal{H}_+, \mathcal{H}_-)$ of Hilbert–Schmidt operators from $\mathcal{H}_+$ to $\mathcal{H}_-$. We are interested in the component of $\mathrm{Gr}(\mathcal{H})$ of virtual dimension zero, i.e., the set of those W for which pr_+ has index equal to zero. As everything can be easly generalized to any virtual dimension, we will not mention the component anymore. The group $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ is relevant here because it acts transitively and holomorphically on $\mathrm{Gr}(\mathcal{H})$. Hence $\mathrm{Gr}(\mathcal{H})$ is the homogeneous space $\mathrm{Gr}(\mathcal{H}) = \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})/G_+$, where G_+ is the group of "upper echelon" operators, namely of operators of the form (1) with $c = 0$. The manifold $\mathrm{Gr}(\mathcal{H})$ is as well a homogeneous space of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$. Indeed, $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ acts transitively on $\mathrm{Gr}(\mathcal{H})$, the stabilizer of $\mathcal{H}_+$ is $\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ and therefore $\mathrm{Gr}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})/\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$.

The previous definition of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ has a rationale. Because of the Hilbert–Schmidt condition one can build a central extension of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$, given by the cocycle

$$\omega_e(A_1, A_2) = \mathrm{tr}(c_1 b_2 - b_1 c_2), \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \tag{2}$$

and extend it to a closed invariant two-form on $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ by setting

$$\omega_g(X_1, X_2) = \omega_e(g^{-1} X_1 g, g^{-1} X_2 g) . \tag{3}$$

This is clearly degenerate and therefore gives a pre-symplectic structure on $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$. To get a symplectic manifold we construct the Marsden–Weinstein reduced manifold by noticing that ω_e is degenerate on the subalgebra $B(\mathcal{H}_+) \times B(\mathcal{H}_-)$ of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$. Accordingly our natural space will be $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})/\mathrm{Gl}(\mathcal{H}_+) \times \mathrm{Gl}(\mathcal{H}_-)$. This is nothing but the complexification of the standard procedure of reducing $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ by $\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ which identifies the restriction of ω to $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ with the Kähler form of the homogeneous manifold $\mathrm{Gr}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})/\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ as in ref. [10].

We can do a bit more work and construct an entire family of symplectic manifolds. For $\zeta \in \mathbb{C}$, consider the family of sets

$$M_\zeta =: \{(W, \phi) \in \mathrm{Gr}(\mathcal{H}) \times \mathrm{gl}_{\mathrm{res}}(\mathcal{H}) \mid \phi(\mathcal{H}) \subseteq W, \ \phi|_W = \zeta \cdot \mathrm{id}_W\} . \tag{4}$$

Clearly enough M_ζ is a subspace of $\mathrm{Gr}(\mathcal{H}) \times \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ with the induced topology. We have projections $\pi_\zeta : M_\zeta \to \mathrm{Gr}(\mathcal{H})$ given by $\pi_\zeta(W, \phi) = W$, with fibre $\pi_\zeta^{-1}(W) = \Im_2(\mathcal{H}/W, W)$. One can easily show (see appendix A) that each M_ζ is actually a fibred manifold over $\mathrm{Gr}(\mathcal{H})$, carrying a $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ action, which is transitive for $\zeta \neq 0$. For an alternative description of M_ζ when $\zeta \neq 0$, consider the set $\tilde{M}_\zeta$ of couples of subspaces (W, W') of $\mathcal{H}$, with $W \in \mathrm{Gr}(\mathcal{H})$ and $W \oplus W' = \mathcal{H}$. The map $\rho : M_\zeta \to \tilde{M}_\zeta$ given by $\rho(W, \phi) = (W, \ker \phi)$ is obviously a bijection. This description makes it clear that there is a canonical section of $M_\zeta \to \mathrm{Gr}(\mathcal{H})$ given by $W \mapsto (W, W^\perp)$, whose image is isomorphic to $\mathrm{Gr}(\mathcal{H})$ thought of as the homogeneous space $U_{\mathrm{res}}(\mathcal{H})/\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$. From proposition 10 we learn that

$$\begin{aligned} M_\zeta &= \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})/\mathrm{Gl}(\mathcal{H}_+) \times \mathrm{Gl}(\mathcal{H}_-), \quad \zeta \neq 0\,, \\ M_0 &= T'^*\mathrm{Gr}(\mathcal{H})\,, \end{aligned} \tag{5}$$

where $T'^*\mathrm{Gr}(\mathcal{H})$ is the holomorphic cotangent bundle. Thus the universal central extension induces a symplectic form, still denoted by ω, on M_ζ (for $\zeta \neq 0$). As shown in appendix A, ω is actually independent of ζ and extends the standard cotangent symplectic structure on M_0. Here we point out few properties which will be relevant in the following:

(i) The action of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}(\mathcal{H})$ is covered by an action of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ on M_ζ which is symplectic; i.e., for any fundamental vector field X_A on M_ζ corresponding to any $A \in \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ we have $\mathcal{L}_{X_A}\omega = 0$.

(ii) For any $A \in \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ there exists a Hamiltonian function, which is explicitly given by ($\zeta \neq 0$)

$$h_A(W, W') = \mathrm{tr}\, A(J_g - J)\,, \tag{6}$$

with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_g = gJg^{-1} \quad \text{if } (W, W') = g(\mathcal{H}_+, \mathcal{H}_-);$$

it is such that $\langle \mathrm{d}h_A, X_B \rangle = \omega(X_A, X_B)$.
The Hamiltonian in (6) is obviously a complexification of that given in ref. [10], whose proof carries over with minor modifications. We notice that $\mathrm{tr}\, A(J_g - J) = \mathrm{tr}\, A((gJ - Jg)g^{-1})$ exists because $[g, J]$ is a trace class operator and the set of all such operators is a two-sided ideal in $B(\mathcal{H})$. Moreover, h_A is $\mathrm{Gl}(\mathcal{H}_+) \times \mathrm{Gl}(\mathcal{H}_-)$ invariant, so it is actually a function on M_ζ which is independent of ζ. Accordingly, the action of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ on M_ζ , $\zeta \neq 0$, is strictly Hamiltonian. On the other hand, the lift to $T'^*\mathrm{Gr}(\mathcal{H})$ of the action of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}(\mathcal{H})$ is strictly Hamiltonian as well, because in such cases there is an equivariant momentum map [1]. Summarizing, we have the following:

Proposition 1. *For any $\zeta \in \mathbb{C}$ the action of $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ on M_ζ is strictly Hamiltonian.* □

Since for $\zeta \neq 0$ all M_ζ are isomorphic, we will drop the suffix and denote by M any of these manifolds, while we will keep the notation M_0 for the cotangent bundle.

3. Integrable phase spaces

Our next task is to look for "maximal" Abelian subalgebras $\mathcal{A} \subset \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ of the form $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \oplus \mathbb{C}\mathbf{1}$ which are such that:

(i) the orbits of the groups $\exp \mathcal{A}_{\pm}$ are isotropic submanifolds of M_ζ;

(ii) the generators of $\mathcal{A}_+$ and $\mathcal{A}_-$ are canonically conjugate with respect to the symplectic structure ω.

By maximal we mean that, if $B \in \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ commutes with all $A \in \mathcal{A}$, then $B \in \mathcal{A}$. So there is no strictly larger Abelian subalgebra of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ containing $\mathcal{A}$.

The orbits of $\exp \mathcal{A}$ through a point $(W, \phi) \in M$ can be identified with integrable phase spaces, $\mathcal{A}_+$ generating the flows along angles and $\mathcal{A}_-$ along actions. Notice that central elements of the form $(\exp \lambda)\mathbf{1}, \lambda \in \mathbb{C}$, act trivially on M. Of course we are interested in studying such $\mathcal{A}$ up to conjugation because the orbit of $\mathcal{A}$ through $g(W, \phi)$ is the same as the orbit of $g\mathcal{A}g^{-1}$ through (W, ϕ). Classifying the conjugacy classes of maximal Abelian subalgebras of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ is itself a problem which we do not attempt to tackle here. Instead we will concentrate on Abelian $\mathbb{C}^*$-algebras which are generated by a normal operator $A \in \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$. In particular we use the fact that $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$, as $B(\mathcal{H})$, is a Banach algebra (with the operator product) and a Banach–Lie algebra (with respect to the commutator of the operator product itself). Thanks to the spectral theorem, all these Abelian algebras are unitarily equivalent to algebras of multiplication operators (although some caution concerning norms is in order here). The main example is obviously the algebra $\gamma \oplus \mathbb{C}\mathbf{1} = \gamma_+ \oplus \gamma_- \oplus \mathbb{C}\mathbf{1}$, which is already represented as an algebra of multiplication operators on $\mathcal{H} = \mathcal{L}^2(\mathrm{S}^1, \mathbb{C})$. Here $\gamma_{\pm}$ is generated by the multiplication operators z^n for $n > 0$ and $n < 0$, respectively. An easy computation shows that $\gamma \oplus \mathbb{C}\mathbf{1}$ is a maximal Abelian subalgebra of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ indeed. Since z is unitary (recall that $|z| = 1$), $\gamma \oplus \mathbb{C}\mathbf{1}$ is the maximal Abelian $\mathbb{C}^*$-algebra generated by z itself. The generic element in γ is therefore a normal operator. Notice that this has no finite-dimensional analogue.

To make contact with a more standard mechanical description, notice that any $A \in \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ defines an endomorphism $A\cdot : \mathrm{gl}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ given by left multiplication and a family of Lie brackets given by

$$[X, Y]_A =: [X, Y] + t\chi_A(X, Y), \tag{7}$$

with $\chi_A(X, Y) =: [AX, Y] + [X, AY] - A[X, Y] = XAY - YAX$ and t a complex number $0 \leq |t| < \|A\|$.

Proposition 2. *The cocycle χ_A defines a trivial deformation of* $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ *and the endomorphism $A\cdot : \mathrm{gl}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ given by left multiplication by A has vanishing (Nijenhuis) torsion.*

Proof. For $0 \leq |t| < \|A\|$, the operator $(1 + tA)$ is invertible and the map $(1 + tA)\cdot : \mathrm{gl}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ given by left multiplication is a linear isomorphism. It is then enough to show that $(1 + tA)[X,Y]_A = [(1 + tA)X, (1 + tA)Y]$. Since χ_A has the special form given above, this requires that $A\chi_A(X,Y) = [AX, AY]$, which is true since $AXAY - AYAB = [AX, AY]$. Accordingly we get

$$\mathcal{N}_A(X,Y) =: A^2[X,Y] + [AX,AY] - A[AX,Y] - A[X,AY] = 0, \tag{8}$$

and $A\cdot$ has vanishing torsion. □

Proposition 2 is actually a quite involved way of stating the trivial fact that the algebra spanned by $A, A^2, A^3, \ldots,$ is Abelian.

From now on we shall stick to the case of γ and study the orbits of $\Gamma = \exp\gamma$ in M. Notice that we can restrict $\gamma \oplus \mathbb{C}\mathbf{1}$ to γ because operators of the form $\lambda\mathbf{1}$ act trivially on M. First recall proposition 10.4.2 of ref. [10]:

Proposition 3. *The action of Γ_- on* $\mathrm{Gr}(\mathcal{H})$ *is free.* □

As for the action of Γ_+ on $\mathrm{Gr}(\mathcal{H})$, let $G_W \subset \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ be the isotropy subgroup of W and set $K_W = \Gamma_+ \cap G_W$.

Proposition 4. *(i) K_W is a normal subgroup of Γ_+. (ii) Any orbit $\mathcal{O}_W^+$ of the group $F_W^+ = \Gamma_+/K_W$ is a complex group. In particular, if it is compact and finite dimensional, J_W is an Abelian variety.*

Proof.

(i) K_W is clearly a group, because it is the intersection of two subgroups and it is trivially normal in Γ_+ because the latter is Abelian.

(ii) It suffices to notice that the action of F_W^+ is free on $\mathcal{O}_W^+$ and then to apply the following lemma.

Lemma 5. *Let $N \subset \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ be the normalizer of Γ_+. Then, for any $g \in N$, there is an isomorphism $\psi : F_W^+ \to F_{gW}^+$.*

Proof. An isomorphism $\tilde{\psi} : \Gamma_+ \to \Gamma_+$ will induce an isomorphism $\psi : F_W^+ \to F_{gW}^+$ if the following diagram

$$\begin{array}{ccc} \Gamma_+ & \xrightarrow{\tilde{\psi}} & \Gamma_+ \\ {\scriptstyle K_W}\big\downarrow & & \big\downarrow{\scriptstyle K_{gW}} \\ F_W^+ & \xrightarrow{\psi} & F_{gW}^+ \end{array}$$

is commutative. It is easy to check that for $g \in N$ the map $\tilde{\psi}(a) = gag^{-1}$ will do the job. □

In particular, the centralizer C of Γ_+ in $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ is such that $\tilde{\psi} = \mathrm{id}$ for any $g \in C$, i.e., $F_W^+ = F_{gW}^+$.

Together with F_W^+ we have another subgroup $F_W^- \subset \Gamma_-$ given by taking the adjoint operators, i.e., $F_W^- = (F_W^+)^*$, and, since Γ_- acts freely, the orbit $\mathcal{O}_W$ of $F_W = F_W^+ \times F_W^-$ through W in $\mathrm{Gr}(\mathcal{H})$ is isomorphic to F_W itself.

Although this orbit is a symplectic manifold with the restriction of the Kähler form on $\mathrm{Gr}(\mathcal{H})$, it is not the right phase space, as the action of F_W^+ is not Hamiltonian. To overcome this drawback, we lift the action to M, where it is clearly Hamiltonian. The orbit $\tilde{\mathcal{O}}_W$ of F_W in M through the point $(W, W^\perp)$ is now a good phase space. So, to get a direct Hamiltonian description we have to enlarge $\mathrm{Gr}(\mathcal{H})$ to M. This is, however, not necessary if one simply wants to see the evolution, because of:

Proposition 6. *The projection $\pi : M \to \mathrm{Gr}(\mathcal{H})$ restricts to an isomorphism $\pi : \tilde{\mathcal{O}}_W \to \mathcal{O}_W$.*

Proof. The action of F_W is free both on $\tilde{\mathcal{O}}_W$ and on $\mathcal{O}_W$. □

We can now be more definite as to the meaning of M as a "universal phase space". We will say that an algebraically integrable system is a Jacobian system if the relevant Abelian varieties are actually Jacobians of curves. In these cases the Krichever map can be given the meaning of a classifying map. Constructing the appropriate category is a task which we leave to the reader. Notice that Jacobian systems always have a Lax representation, as one can immediately imagine by inverting the construction of ref. [6].

4. Real phase spaces and the recursion operator

To accommodate the complexification of the invariant tori of an integrable system in a symplectic framework, one is forced to "complexify" $\mathrm{Gr}(\mathcal{H})$ to M. However, real tori can be recovered in the Grassmannian set up as well.

Recall that M can be considered as the set of couples of subspaces (W, W') of $\mathcal{H}$ which are "comparable" with $(\mathcal{H}_+, \mathcal{H}_-)$. In other words, M is the set of bounded involutions $J_W = gJg^{-1}$, $g \in \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$. On M we have a natural real structure given by

$$(W, W') \mapsto (W'^\perp, W^\perp), \tag{9}$$

or, which is the same, $J_W \mapsto J_W^*$. The set of real points we get in this way is precisely a copy of $\mathrm{Gr}(\mathcal{H})$ canonically embedded in M. It follows that for any W the orbit $\tilde{O}_W$ of F_W intersects the real slice $\mathrm{Gr}(\mathcal{H})$ and the intersection is an orbit of a subgroup of the loop group $\mathrm{LU}(1)$, corresponding to a real form of F_W (as a consequence, it differs from the orbit O_W we started with). In particular, whenever $\tilde{O}_W^+$ is an Abelian variety, $\tilde{O}_W \cap \mathrm{Gr}(\mathcal{H})$ is a real phase space with the restriction of the symplectic structure of M. The picture we get in this way is that, while M is the "universal phase space" for Jacobian systems, $\mathrm{Gr}(\mathcal{H})$ is "universal" for their real sections.

Let us now come back to the Grassmannian $\mathrm{Gr}(\mathcal{H})$. Its holomorphic tangent space $T'_{\mathcal{H}_+}\mathrm{Gr}(\mathcal{H})$ is the space $\mathfrak{I}_2(\mathcal{H}_+;\mathcal{H}_-)$ and is identified with the real tangent space $\mathrm{u}_{\mathrm{res}}(\mathcal{H})/\mathrm{u}(\mathcal{H}_+)\oplus\mathrm{u}(\mathcal{H}_-)$ via the map $X \mapsto \left(\begin{smallmatrix} 0 & -X^* \\ X & 0 \end{smallmatrix}\right)$. The transitive action of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}(\mathcal{H})$ makes it possible to identify $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$-invariant tensors on $\mathrm{Gr}(\mathcal{H})$ with $\mathrm{U}(\mathcal{H}_+)\times\mathrm{U}(\mathcal{H}_-)$-invariant elements in the tensor algebra over $\mathrm{u}_{\mathrm{res}}(\mathcal{H})/\mathrm{u}(\mathcal{H}_+)\oplus\mathrm{u}(\mathcal{H}_-)$ [7]. As is well known, the Kähler structure of $\mathrm{Gr}(\mathcal{H})$ is determined at the base point $\mathcal{H}_+$ by the unique invariant (up to a scalar multiple) inner product $(X,Y)\mapsto g(X,Y) = 2\,\mathrm{tr}(X^*Y)$, together with its imaginary part $(X,Y)\mapsto \Phi(X,Y) = -\mathrm{i}\,\mathrm{tr}(X^*Y - Y^*X)$. This two-form Φ is related to the restriction to $\mathrm{u}_{\mathrm{res}}(\mathcal{H})$ of the universal central extension of $\mathrm{gl}_{\mathrm{res}}(\mathcal{H})$ by

$$\Phi(A_1,A_2) = \mathrm{i}\omega(A_1,A_2) = -\mathrm{i}\,\mathrm{tr}A_1[A_2,J] = -\mathrm{i}\,\mathrm{tr}(c_1^*c_2 - c_2^*c_1),$$
$$A_i = \begin{pmatrix} a_i & -c_i^* \\ c_i & d_i \end{pmatrix},$$

with c_i Hilbert–Schmidt and $a_i^* = -a_i$, $b_i^* = -b_i$ (so that $A_i^* = -A_i$). The value of Φ at the point $W = g\mathcal{H}_+$ on the fundamental vector fields $\hat{\xi},\hat{\eta}$ defined by $\xi,\eta\in\mathrm{u}_{\mathrm{res}}(\mathcal{H})$ is given by

$$\Phi(W;\hat{\xi},\hat{\eta}) = \Phi(g^{-1}\xi g, g^{-1}\eta g) = -\mathrm{i}\,\mathrm{tr}\xi[\eta, J_W]\,. \tag{10}$$

The Hamiltonian function $h_\xi : \mathrm{Gr}(\mathcal{H}) \to \mathbb{R}$ generating the flow on $\mathrm{Gr}(\mathcal{H})$ associated to $\xi\in\mathrm{u}_{\mathrm{res}}(\mathcal{H})$ is given by the real form $h_\xi(W) = -\mathrm{i}\,\mathrm{tr}\xi(J_W - J)$ of eq. (6). Clearly $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ acts on $\mathrm{Gr}(\mathcal{H})$ leaving Φ invariant.

The complex structure on $\mathrm{Gr}(\mathcal{H})$ will be denoted by I; it is a map from vector fields to vector fields with $I^2 = -1$. The dual map with respect to the pairing between vector fields and one-forms will be denoted with I^{T}. We have the standard relation $\omega(X,Y) = g(IX,Y)$, with the property $g(IX,IY) = g(X,Y)$.

The group Γ of the previous section is actually isomorphic to the loop group $\mathrm{L}\mathbb{C}^*/\mathbb{C}$ of maps $\mathrm{S}^1 \to \mathbb{C}^*$ (modulo constants). Accordingly, Γ is the complexification of the loop group $LU(1)/U(1)$ of maps $S^1 \to U(1)$ (modulo constants).

As already mentioned, in the theory of integrable systems one is mainly interested in the action of the two Abelian subgroups of Γ given by $\Gamma_\pm = \exp \gamma_\pm$. In the natural basis $\{z^k, k \in \mathbb{Z}\}$ of $\mathcal{H}$, the matrix of the multiplication operator by z^k, $k > 0$, has the form

$$(\lambda_k)_{ij} = \delta_{i-j,k} = \begin{pmatrix} \lambda_k^{++} & \lambda_k^{+-} \\ 0 & \lambda_k^{--} \end{pmatrix}, \quad i,j \in \mathbb{Z}. \tag{11}$$

The set $\{\lambda_k, k > 0\}$ spans the Lie algebra γ_+. We shall work with the "realification" $\mathrm{lu}(1)_+$ of γ_+. The corresponding generators

$$\begin{aligned} \Lambda_k &=: \lambda_k - (\lambda_k)^* = \left(\delta_{i-j,k} - \delta_{j-i,k}\right)_{i,j\in\mathbf{Z}} \\ &= \begin{pmatrix} \lambda_k^{++} - (\lambda_k^{++})^* & \lambda_k^{+-} \\ -(\lambda_k^{+-})^* & \lambda_k^{--} - (\lambda_k^{--})^* \end{pmatrix}, \quad k > 0, \end{aligned} \tag{12}$$

mutually commute.

The induced fundamental vector fields $\hat{\Lambda}_k$ on $\mathrm{Gr}(\mathcal{H})$, being associated with an Abelian subalgebra of $\mathrm{u}_{\mathrm{res}}(\mathcal{H})$, are Hamiltonian vector fields and the corresponding Hamiltonian functions Poisson commute, $\{h_k, h_l\} = L_{\hat{\Lambda}_k} h_l = \omega(\hat{\Lambda}_k, \hat{\Lambda}_l) = 0$. The set of Hamiltonians $\{h_k, k > 0\}$ are our action variables.

Having a metric on $\mathrm{Gr}(\mathcal{H})$, we can associate a vector field with any one-form. In particular we shall need the vector fields X_k associated with the forms $\mathrm{d}h_k$ and the one-forms Φ_k associated with the vector fields $\hat{\Lambda}_k$.

Proposition 7. *The vector fields X_k defined by $\mathrm{d}h_k(\hat{\eta}) = g(X_k, \hat{\eta})$ and the one-form Φ_k defined by $\Phi_k(\hat{\eta}) = g(\hat{\Lambda}_k, \hat{\eta})$, for any vector fields $\hat{\eta}$, are given by*

$$X_k = I\hat{\Lambda}_k , \tag{13}$$

$$\Phi_k = I^{\mathrm{T}} \mathrm{d}h_k . \tag{14}$$

Proof. By equating $\mathrm{d}h_k(\hat{\eta}) = \omega(\hat{\Lambda}_k, \hat{\eta}) = g(I\hat{\Lambda}_k, \hat{\eta})$ to $g(X_k, \hat{\eta})$ for some vector field X_k, eq. (13) follows. Moreover, from $(I^{\mathrm{T}} \mathrm{d}h_k)(\hat{\eta}) = \mathrm{d}h_k(I\hat{\eta}) = \omega(\hat{\Lambda}_k, I\hat{\eta}) = g(I\hat{\Lambda}_k, I\hat{\eta}) = g(\hat{\Lambda}_k, \hat{\eta})$, eq. (14) follows. □

Notice that the vector fields X_k are not the Hamiltonian vector fields associated with the Hamiltonians h_k but are related to them via the complex structure.

Proposition 8. *The vector fields X_k and the one-forms Φ_k have the following properties:*

$$L_{\hat{\Lambda}_k}\Phi_l = 0\,, \quad k,l > 0\,, \tag{15}$$

$$[\hat{\Lambda}_k, X_l] = 0\,, \quad [X_k, X_l] = 0\,, \quad k,l > 0\,, \tag{16}$$

$$g(\Lambda_k, \Lambda_l) = 4k\delta_{kl}\,, \quad g(X_k, X_l) = 4k\delta_{kl}\,, \quad g(\Lambda_k, X_l) = 0\,, \quad k,l > 0\,, \tag{17}$$

$$\omega(\Lambda_k, \Lambda_l) = 0\,, \quad \omega(X_k, X_l) = 0\,, \quad \omega(\Lambda_k, X_l) = 4k\delta_{kl}\,, \quad k,l > 0\,. \tag{18}$$

Proof. Equation (15) follows from the invariance of the complex structure and the fact that the Hamiltonians h_k mutually commute. Equation (16) follows from the invariance of the complex structure, the vanishing of the torsion of the complex structure and the vanishing of the commutator of any two $\hat{\Lambda}_k$. Finally, simple computations give (17) and (18). □

We are now ready to define the recursion operator for the action of $\mathrm{LU}(1)^+$ on $\mathrm{Gr}(\mathcal{H})$ We first recall some basic fact about recursion operators and their use in the theory of integrable systems [5].

Let us consider a dynamical system on a manifold P and denote by Δ the vector field on P which generates the dynamical evolution. Assume that there is a (1,1) tensor field R on P which is invariant under the dynamics,

$$L_\Delta R = 0\,. \tag{19}$$

We shall denote with the same symbol the endomorphism of $\mathcal{X}(P)$ (vector fields on P) and of its dual $\mathcal{X}^*(P)$ (one-forms on P) associated with any (1,1) tensor field and defined by

$$\langle R(X), \theta\rangle \equiv \langle X, R(\theta)\rangle \equiv R(X,\theta)\,, \quad \forall X \in \mathcal{X}(P),\ \theta \in \mathcal{X}^*(P).$$

Any Δ-invariant tensor R will map Δ-invariant vector fields into Δ-invariant ones. Iterating R, one generates an algebra $\mathcal{A}_+$ of vector fields all commuting with Δ,

$$\mathcal{A}_+ = \{\Delta, R(\Delta), R^2(\Delta), \ldots, R^k(\Delta), \ldots\}\,. \tag{20}$$

The commutation relations of this algebra are expressed in terms of R and of its Nijenhuis tensor $\mathcal{N}_R$ defined as in (8). Then condition $\mathcal{N}_R = 0$ together with (19) implies that the algebra (20) is Abelian. In addition, if R has at least two eigenvalues, the dynamics Δ separates in dynamics of lower dimension. This is easily seen in the case of a diagonalizable R, although this hypothesis can be relaxed. Let $\{e_m\}$ be its eigenvectors with corresponding eigenvalues

$\{\mu_m\}$, $Re_m = \mu_m e_m$. Then the vanishing of the tensor $\mathcal{N}_R$ is equivalent to the conditions

$$(\mu_m - \mu_n) L_{e_m} \mu_n = 0 \,, \tag{21}$$

$$(R - \mu_m) \circ (R - \mu_n)([e_m, e_n]) = 0 \,. \tag{22}$$

Condition (22) tells us that the frame $\{e_m\}$ is holonomic. Indeed, if $\{\theta^m\}$ are dual covectors, $\langle \theta^m, e_n \rangle = \delta^m_n$, then $R\theta^m = \mu_m \theta^m$. By contracting (22) with θ^l one gets

$$(\mu_l - \mu_m)(\mu_l - \mu_n)\langle \theta^l, [e_m, e_n] \rangle = 0 \,, \tag{23}$$

and therefore the frame $\{e_m\}$ is holonomic. As a consequence of the condition $\mathcal{N}_R = 0$, the diagonal form of R is

$$R = \sum_m \mu_m e_m \otimes \theta^m \,. \tag{24}$$

It is now easy to see that the invariance condition (19) for a diagonal R as in (24) implies both the separability of Δ in lower-dimensional dynamics and the fact that the eigenvalues μ are constants of the motion for Δ. Indeed, by acting with Δ on both sides of $Re_m = \mu_m e_m$ and by contracting with θ^n and using (23), one gets

$$L_\Delta \mu_n = 0 \,, \tag{25}$$

$$(\mu_m - \mu_n)\, L_{e_n} \langle \theta^m, \Delta \rangle = 0 \,, \quad m \neq n \,, \tag{26}$$

which state that the eigenvalues are constant and that Δ is separable, respectively.

When additional conditions are required on the spectrum of R, one can sharpen the previous results on separability. For instance, if each eigenvalue μ is doubly degenerate without stationary points, namely $\mathrm{d}\mu \neq 0$, the dynamics separates in a sum of two-dimensional dynamics and there is a constant of the motion for each of them. Therefore the system is completely integrable. The dynamics Δ is not supposed to be Hamiltonian. However, it turns out that by using the hypothesis that the eigenspaces of R are bidimensional and the fact that $\mathrm{d}\mu \neq 0$, one can construct a Hamiltonian structure with respect to which Δ is Hamiltonian. It separates in one-degree of freedom dynamics which are Hamiltonian and completely integrable.

We propose the following (1,1) tensor field for the action of $\mathrm{LU}(1)^+$ on $\mathrm{Gr}(\mathcal{H})$:

$$\begin{aligned} R &= \sum_{k>0} \frac{h_k}{4k} \{ X_k \otimes \mathrm{d}h_k + \hat{\Lambda}_k \otimes \Phi_k \} \\ &= \sum_{k>0} \frac{h_k}{4k} \{ I\hat{\Lambda}_k \otimes \mathrm{d}h_k + \hat{\Lambda}_k \otimes I^{\mathrm{T}}\, \mathrm{d}h_k \} \,. \end{aligned} \tag{27}$$

The tensor R can also be written as

$$R(\,\cdot\,) = \sum_{k>0} \frac{h_k}{4k}\{X_k \otimes g(X_k,\,\cdot\,) + \hat{A}_k \otimes g(\hat{A}_k,\,\cdot\,)\}$$
$$= \sum_{k>0} \frac{h_k}{4k}\{I\hat{A}_k \otimes g(I\hat{A}_k,\,\cdot\,) + \hat{A}_k \otimes g(\hat{A}_k,\,\cdot\,)\}\,. \tag{28}$$

Proposition 9. *The tensor R has the following properties.*

(i) It exists.

(ii) It is degenerate. Indeed, it vanishes on the orthogonal complement of (the closure of) $\mathrm{span}\{\bigoplus_k \hat{A}_k \bigoplus_l X_l\}$. *On the other hand,*

$$R(\hat{A}_k) = h_k\hat{A}_k\,, \qquad R(X_k) = h_k X_k\,, \quad k>0\,, \tag{29}$$

so that the remaining "eigenspaces" of R are two-dimensional at each point.

(iii) It is invariant along any vector field $\hat{A}_k$,

$$L_{\hat{A}_k} R = 0\,, \qquad k>0\,. \tag{30}$$

(iv) It has vanishing torsion,

$$\mathcal{N}_R = 0\,. \tag{31}$$

Proof.

(i) For any point in $\mathrm{Gr}(\mathcal{H})$ eq. (17) implies that the two families $\{X_k/\sqrt{2k}\}$ and $\{\hat{A}_k/\sqrt{2k}\}$ are orthonormal families of vectors in a Hilbert space (of Hilbert–Schmidt operators). Therefore the existence of R is equivalent to the boundedness of the family $\{h_k\}$ [11]. This can be easily checked at any point in $\mathrm{Gr}(\mathcal{H})$, and it implies the pointwise existence of R.

(ii) Obvious.

(iii) Just compute using properties of the $\hat{A}_k$, X_k and ϕ_k .

(iv) As we have seen before, the vanishing of $\mathcal{N}_R$ is equivalent to conditions (21) and (22). Now condition (21) is easily verified. When the eigenvalue $\mu_k = 0$, it is trivial. When $e = \hat{A}_k$, it follows from the fact that the Hamiltonians h_k mutually commute; when $e = X_k$, it follows from (17); when e is in the orthogonal complement of $\mathrm{span}\{\bigoplus_k \hat{A}_k \bigoplus_l X_l\}$, it follows from the orthogonality condition. Let us now analyze (22). When the two e's are either both a $\hat{A}$ or both an X or one is a $\hat{A}$ and the other an X, (22) follows from the vanishing of the corresponding commutators. If e_m is in the the orthogonal complement of $\mathrm{span}\{\bigoplus_k \hat{A}_k \bigoplus_l X_l\}$ and e_n is either $\hat{A}_k$ or X_k, then the invariance of the metric g implies that the commutators $[e_m, \hat{A}_k]$ and $[e_m, X_k]$ are both in the orthogonal complement of $\mathrm{span}\{\bigoplus_k \hat{A}_k \bigoplus_l X_l\}$, so that they are eigenvectors of R corresponding to the eigenvalue zero; (22) then follows. Finally, let us take e_m and e_n both in the kernel of R. Then $R([e_m, e_n]) = -(L_{e_m}R)(e_n) = 0$, since from the explicit form (27) or (28) of R it follows that the kernel of $L_{e_m}R$ is the same as that of R. This completes the proof of (31).

5. Concluding remarks

Although most of the results of this paper are already available in the literature, we felt that an explicit link with symplectic dynamics was missing. Giving it provides new tools to tackle some interesting questions in the realm of integrable systems which we want to discuss here.

First of all there is the problem of studying relations between integrable systems and Lax systems. The construction of Griffiths [6] explicitly realizes the Lax flow as a flow N_t on the Jacobian of the spectral curve. By mapping to $\mathrm{Gr}(\mathcal{H})$ via the Krichever map, one finds a Hamiltonian system which is completely integrable whenever the flow N_t is linear. Conversely, given a Jacobian system, we can reconstruct a Lax representation for it by choosing a representation of the Krichever curve C as a branched covering of $\mathbb{P}^1$. Given an ample initial datum L_0, the flow of Γ^+ on $\mathrm{Jac}(C)$ gives a family $L_t = L_0 \otimes N_t$ of ample line bundles on C. Inverting the eigenvector map of ref. [6], one gets a Lax matrix $\mathbf{L} = \mathbf{L}(t)$ and a "Hamiltonian" $\mathbf{B}$ such that the flow L_t has equation $\dot{\mathbf{L}} = [\mathbf{B}, \mathbf{L}]$. Although filling in all the details of this picture requires some work, one immediately realizes that the Lax form of a given Jacobian system is not unique since the representation $C \to \mathbb{P}^1$ is not. In other words, the explicit dependence of $\mathbf{L}$ on the spectral parameter as well the rank of $\mathbf{L}$ are not unique. One may profit from this ambiguity in circumventing the Griffiths obstruction by suitably changing the representation of the spectral curve. This might also help in understanding the "right" way of inserting the spectral parameter in the Lax matrix. Incidentally, we notice that the evolution equations on M naturally have a Lax form. Indeed, if $g(t) = \exp tB$ and $J_h = hJh^{-1}$, then the flow induced by B reads $J_{g(t)\cdot h} = g(t) J_h g(t)^{-1}$ and satisfies the equation $\dot{J} = [B, J_h]$.

Secondly, the present set up makes less mysterious the role of recursion operators in integrable systems. Indeed, all Jacobian systems have a natural recursion operator R, as shown in sections 3 and 4. Although constructing R in explicit examples may be as hard as finding action–angle coordinates, the way to do it is in principle clear and works more generally for the set up of ref. [5]. Given a dynamical vector field Δ, one symply needs to realize the Abelian algebra generated from Δ by successive applications of R, as an Abelian $\mathbb{C}^*$-algebra in $B(\mathcal{H})$ and then apply the spectral theorem. Notice that we do not need any compatibility condition between R and the symplectic structure on $\mathrm{Gr}(\mathcal{H})$, although these might be automatically satisfied when restricting to an orbit $\mathcal{O}_W$.

As a final comment, we notice that the manifold M given in (4) can be considered as a parameter space for solutions of the modified Yang–Baxter equation. Indeed any point in M corresponds to an involution of the form $R_g = gJg^{-1}$, for $g \in \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$, which satisfies both $\mathcal{N}_{R_g} = 0$ and $R_g^2 = 1$. These two conditions are actually equivalent to the modified Yang–Baxter equation [13]. Although these R's are of little use as recursion operators, they may turn out to be relevant for the understanding of the geometrical meaning of the Yang–Baxter r-matrices. Indeed, any $g \in F_W^+$ implements a translation t_g along any orbit $\tilde{\mathcal{O}}_W$. When these are Jacobians of curves, the restriction of the line bundle Det* coincides with the Θ-bundle on the Jacobian itself and the action of g on Det* gives a translation $t_g^* \Theta$. From the theorem of the square (see, e.g., ref. [2]) we have an isomorphism $\psi : t_{a_1+a_2}^* \Theta \otimes \Theta \to t_{a_1}^* \Theta \otimes t_{a_2}^* \Theta$. Explicit representations of the isomorphism ψ can be directly related to quantum Yang–Baxter matrices [3].

Appendix A. A family of symplectic manifolds

The family $\mathcal{M} = \bigcup_\zeta M_\zeta$, with M_ζ given by (4), is actually a fibred manifold over $\mathbb{C} \times \mathrm{Gr}(\mathcal{H})$ which can be covered by open sets of the form $V_S = \mathbb{C} \times U_S \times \Im_2(\mathcal{H}_{S'}, \mathcal{H}_S)$ indexed by Dirac seas $S \in \mathbb{Z}$ (with $S' = \mathbb{Z} - S$). Here $U_S \subset \mathrm{Gr}(\mathcal{H})$ is the open set of W's with w_S invertible, if we think of $W \in \mathrm{Gr}(\mathcal{H})$ as the image of a map

$$\begin{pmatrix} w_S \\ w_{S'} \end{pmatrix} : \mathcal{H}_S \to \mathcal{H},$$

and $\Im_2(\mathcal{H}_{S'}, \mathcal{H}_S)$ is the space of Hilbert–Schmidt operators $b : \mathcal{H}/\mathcal{H}_S = \mathcal{H}_{S'} \to \mathcal{H}_S$. Indeed we have homeomorphisms

$$\pi^{-1}(\mathbb{C} \times U_S) \to V_S \,, \quad (\zeta, W, \phi) \mapsto (\zeta, T_S, b_S) \,, \tag{A.1}$$

where

$$T_S = w_{S'} w_S^{-1}, \qquad b_S = \begin{pmatrix} 1 & 0 \\ -T_S & 1 \end{pmatrix} \phi \begin{pmatrix} 1 & 0 \\ T_S & 1 \end{pmatrix} \Bigg|_{\mathcal{H}_{S'}} .$$

In fact, any $W \in U_S$ can be written as $W = g\mathcal{H}_S$ with

$$g = \begin{pmatrix} 1 & 0 \\ T_S & 1 \end{pmatrix},$$

and hence

$$\begin{pmatrix} 1 & 0 \\ -T_S & 1 \end{pmatrix} \phi \begin{pmatrix} 1 & 0 \\ T_S & 1 \end{pmatrix}$$

has the form $\left(\begin{smallmatrix}\zeta 1 & b_S\\ 0 & 0\end{smallmatrix}\right)$ relative to the decomposition $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_{S'}$. Notice that

$$\begin{pmatrix} 1 & 0 \\ T_S & 1 \end{pmatrix} : \mathcal{H}_S \oplus \mathcal{H}_{S'} \to \mathcal{H}$$

is a map with image $W \oplus \mathcal{H}_{S'}$.

Whenever $W \in U_{S_0} \cap U_{S_1}$, there is a linear isomorphism

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$$

such that the diagram

$$\begin{array}{ccc} \mathcal{H}_{S_0} \oplus \mathcal{H}_{S_0'} & \xrightarrow{A} & \mathcal{H}_{S_1} \oplus \mathcal{H}_{S_1'} \\ {\scriptstyle \hat{T}_0}\downarrow & & \downarrow{\scriptstyle \hat{T}_1} \\ \mathcal{H} & \xrightarrow{B} & \mathcal{H} \end{array} \tag{A.2}$$

commutes, where $B = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ is the matrix of the identity transformation $\mathcal{H}_{S_0} \oplus \mathcal{H}_{S_0'} \to \mathcal{H}_{S_1} \oplus \mathcal{H}_{S_1'}$ (i.e., $a : \mathcal{H}_0 \to \mathcal{H}_1$, etc.) and

$$\hat{T}_i = \begin{pmatrix} 1 & 0 \\ T_{S_i} & 1 \end{pmatrix}, \quad i = 0, 1.$$

The commutativity of the diagram (A.2) implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T_{S_0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T_{S_1} & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

In turn one gets

$$T_{S_1} = (c + dT_{S_0})(a + bT_{S_0})^{-1}, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a + bT_{S_0} & b \\ 0 & d - T_{S_1} b \end{pmatrix},$$

the first relation being the usual coordinate transformation on $\mathrm{Gr}(\mathcal{H})$, which shows that T_{S_1} is a holomorphic function of T_{S_0} in the open set where $a + bT_{S_0}$ is invertible. With these data we can identify the corresponding b_{S_i}. We have that

$$\phi_{S_1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi_{S_0} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1},$$

and therefore, by setting

$$\phi_{S_i} = \begin{pmatrix} 1 & 0 \\ -T_S & 1 \end{pmatrix} \begin{pmatrix} \zeta 1 & b_{S_i} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T_S & 1 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \zeta 1 & b_{S_1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \zeta 1 & b_{S_0} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\beta\delta^{-1} \\ 0 & \delta^{-1} \end{pmatrix},$$

from which $b_{S_1} = -\zeta\beta\delta^{-1} + \alpha b_{S_0}\delta^{-1}$.

Summarizing, the coordinate transformations on $\mathcal{M}$ are given by

$$\begin{aligned} V_{S_0} \to V_{S_1}, \quad & (\zeta, T_{S_0}, b_{S_0}) \mapsto (\zeta, T_{S_1}, b_{S_1}) \\ & T_{S_1} = (c + dT_{S_0})(a + bT_{S_0})^{-1}, \\ & b_{S_1} = (-\zeta b + (a + bT_{S_0})b_{S_0})(d - T_{S_1}b)^{-1}. \end{aligned} \tag{A.3}$$

Notice that for $\zeta \neq 0$ the clutching functions for the b_{S_i} are not homogeneous, and therefore M_ζ is not a vector bundle. For $\zeta = 0$ instead, they become homogeneous and M_0 is isomorphic to the $(1,0)$-cotangent bundle $T'^{*}\mathrm{Gr}(\mathcal{H})$ of $\mathrm{Gr}(\mathcal{H})$. This easily follows from the exact sequence

$$0 \to \mathcal{S} \to \mathrm{Gr}(\mathcal{H}) \times H \to \mathcal{Q} \to 0, \tag{A.4}$$

$\mathcal{S} = \{(W, f) \in \mathrm{Gr}(\mathcal{H}) \times \mathcal{H} \mid f \in W\}$ being the tautological bundle. In analogy with the finite-dimensional case we have that $T'\mathrm{Gr}(\mathcal{H}) = \Im_2(\mathcal{S}, \mathcal{Q})$ and $T'^{*}\mathrm{Gr}(\mathcal{H}) = \Im_2(\mathcal{Q}, \mathcal{S})$, where $\Im_2$ means Hilbert–Schmidt homomorphisms. Accordingly, the transition functions read $x_{S_1} = \delta x_{S_0}\alpha^{-1}$ for $T'\mathrm{Gr}(\mathcal{H})$ and $b_{S_1} = \alpha b_{S_0}\delta^{-1}$ for $T'^{*}\mathrm{Gr}(\mathcal{H})$.

We next construct a relative symplectic structure on $\mathcal{M}$. On each open set V_S define a one-form θ_S by setting $\theta_S(X) = \mathrm{tr}\, x b_S$ for $X = \left(\begin{smallmatrix} 0 & x' \\ x & 0 \end{smallmatrix}\right)$ a tangent vector field on V_S. This is global for $\zeta = 0$; indeed,

$$\begin{aligned} \mathrm{tr}\, x_{S_1} b_{S_1} &= \mathrm{tr}\, \delta x_{S_0}\alpha^{-1}(-\zeta b + \alpha b_{S_0})\delta^{-1} \\ &= \mathrm{tr}\, x_{S_0} b_{S_0} - \zeta\, \mathrm{tr}\, x_{S_0}\alpha^{-1} b, \end{aligned} \tag{A.5}$$

the difference being $\theta_{S_0S_1} = -\zeta\, \mathrm{tr}\, x_{S_0}\alpha^{-1}b$. The family of two-forms $\mathrm{d}\theta|_{V_S}$ merges to a global closed two-form ω on $\mathcal{M}$ if and only if $\mathrm{d}\theta_{S_0S_1} = 0$. Since b is constant, one easily computes

$$\mathrm{d}\theta_{S_0S_1}(X, Y) = \mathrm{tr}(x\langle D\alpha^{-1}, y\rangle - y\langle D\alpha^{-1}, x\rangle)b,$$

where $\langle DA, B\rangle$ denotes the Frechet derivative of A along B. Using the fact that $\langle D\alpha^{-1}, y\rangle = -\alpha^{-1}\langle D\alpha, y\rangle\alpha^{-1}$ and $\langle D\alpha, y\rangle = by$,

$$\mathrm{d}\theta_{S_0S_1}(X, Y) = \mathrm{tr}\, x\alpha^{-1}by\alpha^{-1}b - \mathrm{tr}\, y\alpha^{-1}bx\alpha^{-1}b = 0.$$

Next we compute $\omega|_{V_S} = \mathrm{d}\theta_S$ by noticing that $L_X\theta(Y) = \mathrm{tr}\langle DY, X\rangle + \mathrm{tr}\, yx'$. After some algebra we find $\omega|_{V_S}(X, Y) = \mathrm{tr}(yx' - xy')$, which is obviously globally defined, closed and non-degenerate. Summarizing we have

Proposition 10. *(1) For $\zeta = 0$, $\omega|_{M_0}$ is the standard cotangent symplectic form on $M_0 = T'^*\mathrm{Gr}(\mathcal{H})$; (2) for any $\zeta \neq 0$, $\omega|_{M_\zeta}$ coincides with the restriction of the central extension of* $\mathrm{Gl}_{\mathrm{res}}(\mathcal{H})$ *to* $M_\zeta = \mathrm{Gl}_{\mathrm{res}}(\mathcal{H})/\mathrm{Gl}(\mathcal{H}_+) \times \mathrm{Gl}(\mathcal{H}_-)$. □

Proposition 11. *The family $(\mathcal{M}, \omega)$ is a deformation of the cotangent symplectic structure on the central fibre $M_0 = T'^*\mathrm{Gr}(\mathcal{H})$.* □

Appendix B. The example of the harmonic oscillator

The subset $\mathbb{C}^* = \mathbb{C} - \{0\}$ of the phase space of the harmonic oscillator (where the Hamiltonian $H = \frac{1}{2}zz^*$, $z = p + iq$, is not critical) can be mapped to $\mathrm{Sl}(2, \mathbb{C})$ by

$$L : \mathbb{C}^* \to \mathrm{Sl}(2, \mathbb{C}) \ , z \mapsto \frac{1}{\sqrt{2H}} \begin{pmatrix} p & q \\ q & -p \end{pmatrix} . \tag{B.1}$$

The Hamiltonian flow on $\mathbb{C}^*$ is then equivalent to a Lax equation of the form $\dot{L} = [L, B]$, with $B = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ [4]. We extend this embedding to the complexification $\mathbb{C}^* - \{0\}$ of the real phase space by letting p and q be complex. Now $L : \mathbb{C}^2 \to \mathbb{C}^2$ is an involution, i.e., $L^2 = \mathbf{1}$ and the solution of $\dot{L} = [L, B]$ reads

$$L(t) = \mathrm{e}^{+tB} L(0)\, \mathrm{e}^{-tB} \ . \tag{B.2}$$

This is really a flow on the space of involutions on $\mathbb{C}^2$ similar to, e.g.,

$$L(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(initial condition for $q = 0$, $p = p_0 \neq 0$). Such a space is the finite-dimensional analogue of M. The analogue of γ_+ , γ_- are the one-dimensional subalgebras spanned by

$$B_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad B_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

which, however, do not commute, a marked difference with the infinite-dimensional case. The Lax pair $L_\xi = \sqrt{2H}(L + \xi B)$, B gives equivalent Lax equations. The associated spectral curve is the locus $C = \{(\lambda, \xi) \mid \det(L_\xi - \lambda \mathbf{1})\}$, i.e., $\lambda^2 + \xi^2 = 2H$. This is a copy of $\mathbb{P}^1$ with two points removed, as the condition that H is non-degenerate easily implies. These two points can be actually identified, giving us a node curve of genus 1. Its Jacobian is the locus $p^2 + q^2 = 2H$, $H \neq 0$.

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